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Midcourse Guidance for a Short-Range Attack Missile Using Error Compensation

Dinesh V. Muzumdar* and David G. Hull†
University of Texas at Austin, Austin, Texas 78712

Midcourse guidance laws are developed for a short-range, bank-to-turn attack missile with a high thrust-to-weight ratio. The trajectory optimization formulation is to minimize the flight time from a low-altitude, subsonic launch to a fixed target 100–300 n mile from the launch aircraft at any azimuth. After the thrust phase, the missile coasts in a vertical plane at zero angle of attack. For guidance, optimal guidance is selected, and two guidance laws are developed by formulating an equivalent optimal control problem over the thrust phase. The analytical guidance law is obtained by making approximations in the optimal control problem. It performs well for midcourse guidance but needs terminal guidance to hit the target. The error compensation (EC) guidance law is obtained by replacing the approximation terms by bounded controls, where the bounds are handled indirectly by adding penalty terms to the performance index. The EC weights are determined by using the EC control in the trajectory optimization problem and minimizing the flight time with respect to the weights. On board, weights can be stored for an array of target ranges and azimuths and obtained at launch by table lookup. The EC guidance law enables the missile to hit the target without terminal guidance.

Nomenclature

A_T	= target azimuth, deg
C_D	= drag coefficient
C_{D0}	= zero-lift drag coefficient
C_L	= lift coefficient
$C_{L\alpha}$	= lift-curve slope, rad^{-1}
D	= drag, lb
e	= error compensation terms
g	= gravitational acceleration, ft/s^2
K	= induced drag factor
L	= lift, lb
M	= Mach number; error compensation weights
m	= mass, slugs
R_T	= target range, n mile
t_f	= total flight time, s
u, v, w	= inertial velocity components, ft/s
V	= velocity, ft/s
x, y, h	= inertial position components, ft
α	= angle of attack, rad
β	= propellant mass flow rate, slugs/s
γ	= flight-path angle, rad
θ	= thrust inclination angle, rad
λ	= Lagrange multiplier function
μ	= bank angle, rad
ν	= Lagrange multiplier constant
ϕ	= thrust azimuth angle, rad
ψ	= heading angle, rad

Subscripts

bo	= burnout
f	= final
T	= target
0	= initial or sample

Introduction

A SHORT-RANGE attack missile (SRAM) is a beyond-visual-range, bank-to-turn missile with a high thrust-to-weight ratio. Its mission is assumed to be the following: the missile is launched subsonically at low altitude from an aircraft; it performs a thrusting, climbing turn to point toward the target at burnout; and it coasts in a vertical plane at zero angle of attack to a fixed target. The target can be located between 100 and 300 n mile from the launch aircraft and at any point on the compass.

The approach taken here for the development of midcourse guidance laws is to find an approximate analytical optimal control from the current sample point to the desired final conditions, compute the optimal control at the sample point, and hold it constant over the sample period (sample and hold).

A standard method for finding an approximate analytical solution of an optimal control problem is perturbation theory.[1,2] Here, a small parameter is identified; the functional form of the solution is expanded in a Taylor series in the small parameter; and equations are identified for obtaining the zero-, first-, . . . , and higher-order solutions. The zero-order solution corresponds to the approximate analytical solution, and the first-order solution (hopefully analytical) improves the performance of the approximate solution.

In this paper, a technique called error compensation[3] is employed. Error compensation (EC) does not require a small parameter. The approximation terms in the differential equations and the boundary conditions are treated as bounded control variables, and the bounds are handled indirectly by adding penalty terms to the performance index. Weights associated with these controls are then used to tune the EC controls to the true optimal control problem.

To begin the study, true optimal trajectories are computed to determine what approximations can be made. Because of missile vunerability, the total flight time is minimized. Then, an approximate analytical optimal control[4] is obtained by formulating an equivalent optimal control problem over the thrust phase. Aerodynamic terms are neglected in the equations of motion, and burnout conditions are created for hitting the target at the end of the coast. Next, the analytical guidance law is improved by adding error compensation terms to the equations of motion and the burnout conditions. The analytical and EC optimal control problems are formulated simultaneously. Finally, the performances of these two guidance laws are compared.

This paper makes three contributions. First, an analytical guidance law is presented for a SRAM. It works well for midcourse guidance but needs terminal guidance to hit the target. Second, the analytical guidance law is improved by including error

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*Graduate Research Assistant, Department of Aerospace Engineering and Engineering Mechanics.

†M. J. Thompson Regents Professor, Department of Aerospace Engineering and Engineering Mechanics. Associate Fellow AIAA.

compensation terms. The EC guidance law enables the missile to hit the target without terminal guidance. Third, this is the first time that an EC term is used in an approximate boundary condition.

It is emphasized that this is a feasibility study and that many of the details associated with an actual implementation are not considered.

True Optimal Trajectories

In analytical guidance-law development, the first step is to compute true optimal trajectories and determine what approximations can be made to simplify the optimal control problem. The physical characteristics of the missile used in this study are presented in Appendix A.

Tactical considerations and missile vulnerability dictate that the mission be completed in the shortest possible time. Therefore, the SRAM optimal control problem can be stated as follows: Find the control histories $\alpha(t)$ and $\mu(t)$ that minimize the flight time

$$J = t_f \tag{1}$$

to go from the prescribed initial (launch) conditions

$$t_0 = 0\ \text{s},\quad x_0 = 0\ \text{ft},\quad y_0 = 0\ \text{ft},\quad h_0 = 1000\ \text{ft}$$

$$u_0 = 1000\ \text{ft/s},\quad v_0 = 0\ \text{ft/s},\quad w_0 = 0\ \text{ft/s} \tag{2}$$

$$m_0 = 68.3\ \text{slugs}$$

to the prescribed burnout condition

$$t_{bo} = 20.3\ \text{s} \tag{3}$$

to the prescribed final conditions

$$x_f = x_T,\qquad y_f = y_T,\qquad h_f = h_T \tag{4}$$

which represent the fixed target. The differential constraints are the missile equations of motion, and an inequality constraint is imposed on the angle of attack.

If the flat-Earth model is assumed, the equations of motion for the thrust phase of a bank-to-turn missile modeled as a point mass can be written as[5]

$$\dot{x} = u \tag{5}$$

$$\dot{y} = v \tag{6}$$

$$\dot{h} = w \tag{7}$$

$$\dot{u} = (1/m)[T\cos\theta\cos\phi - D\cos\gamma\cos\psi - L(\cos\mu\sin\gamma\cos\psi + \sin\mu\sin\psi)] \tag{8}$$

$$\dot{v} = (1/m)[T\cos\theta\sin\phi - D\cos\gamma\sin\psi - L(\cos\mu\sin\gamma\sin\psi - \sin\mu\cos\psi)] \tag{9}$$

$$\dot{w} = (1/m)(T\sin\theta - D\sin\gamma - mg + L\cos\mu\cos\gamma) \tag{10}$$

$$\dot{m} = -\beta \tag{11}$$

The azimuth and inclination of the thrust vector (missile centerline) relative to the inertial axes are given by

$$\theta = \arcsin(\sin\gamma\cos\alpha + \sin\alpha\cos\mu\cos\gamma) \tag{12}$$

$$\phi = \arccos(\cos\gamma\cos\psi\cos\alpha - \sin\alpha\sin\mu\sin\psi - \sin\alpha\cos\mu\sin\gamma\cos\psi) \tag{13}$$

where the flight-path inclination and heading angle are written as

$$\gamma = \arcsin\frac{w}{\sqrt{u^2+v^2+w^2}} \tag{14}$$

$$\psi = \arccos\frac{u}{\sqrt{u^2+v^2}} \tag{15}$$

Table 1 True optimal results

Test case no.	Target range, n mile	Target azimuth, deg	γ_{bo}, deg	V_{bo}, ft/s	h_{bo}, ft	Final time, s	V_f, ft/s
1	100	0	10.3	8279	48,962	145.6	974
2	100	60	10.9	8133	48,801	147.7	978
3	100	120	12.8	7763	48,238	153.8	990
4	100	180	14.1	7245	51,056	158.1	1005
5	200	0	19.3	8634	51,740	207.0	1322
6	200	60	20.3	8487	50,990	210.5	1360
7	200	120	22.6	8117	50,985	219.3	1466
8	200	180	25.5	7480	52,317	230.0	1607
9	300	0	28.0	8862	54,650	276.9	2393
10	300	60	25.1	8725	54,436	283.3	2488
11	300	120	32.5	8449	55,321	299.0	2697
12	300	180	37.7	8101	58,199	325.6	3034

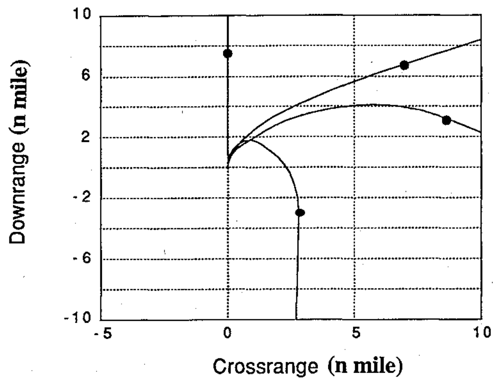

Fig. 1 True optimal trajectories, 100 n mile.

The thrust and the propellant mass flow rate are assumed constant. The aerodynamic forces satisfy the relations

$$D = \tfrac{1}{2}C_D\rho SV^2,\qquad L = \tfrac{1}{2}C_L\rho SV^2 \tag{16}$$

where an exponential atmosphere[5] is used to compute the density, and the speed of sound is assumed to be 1000 ft/s. The force coefficients are modeled as

$$C_L = C_{L\alpha}(M)\alpha \tag{17}$$

$$C_D = C_{D0}(M) + K(M)C_L^2 \tag{18}$$

Hence, the drag and the lift are functions of the altitude, the velocity, and the angle of attack. Finally, the angle of attack is limited to 15 deg.

At t_{bo}, the engines shut off, and the vehicle glides at zero angle of attack and zero bank angle. Therefore, in Eqs. (5) through (10),

$$T = L = \alpha = \mu = 0,\qquad t > t_{bo} \tag{19}$$

The solution of this optimal control problem, referred to as the true optimal solution, is obtained using piecewise linear controls and nonlinear programming. Table 1 is a summary of the true optimal results. It indicates that the final velocity increases with target range and azimuth. This may be counterintuitive, but the optimal control problem is to minimize the final time. Hence, whatever time is lost while making the turn can be made up by higher velocities during the coast. Note that the final time behaves as expected. In all cases, the miss distance is less than 10^{-2} n mile.

Ground tracks for cases 1–4 near launch are shown in Fig. 1. The missile turns rapidly toward the target, and after burnout (solid dot), the trajectory goes straight toward the target. Note that the thrust phase covers less that 10% of the total distance.

EC Optimal Control Problem

The EC optimal control problem contains the analytical optimal control problem. The latter is obtained by discarding the EC terms.

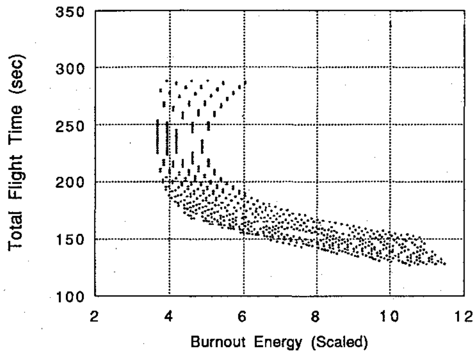

Fig. 2 Coast time vs burnout energy, 100 n mile.

For real-time guidance of a SRAM, an analytical solution of the optimal control problem is desired. Since there is no analytical solution for the coast at zero angle of attack,[6] the optimal control problem must be formulated for the thrust phase only. An analytical optimal control exists for the problem of maximizing the burnout velocity of a missile in a vacuum provided the performance index and final conditions are functions of the burnout velocity components only. In this section, it is shown that the SRAM problem can be made to fit into this category by making suitable approximations. First, it is shown that minimizing the total flight time is equivalent to maximizing the burnout velocity. Next, the ballistic coast is analyzed to determine the burnout conditions that must be satisfied so that at the end of the coast the missile hits the target. Then, the missile equations of motion are simplified. Finally, the EC optimal control problem is formulated. The angle-of-attack inequality constraint is not imposed. If optimal value of α exceeds 15 deg, $\alpha = 15$ deg is used.

For known values of burnout altitude, burnout velocity, and burnout flight-path angle, the equations of motion for planar ballistic coast can be integrated numerically to the final altitude to obtain the coast range and coast time. This process can be used iteratively to determine the relationship between the coast time and burnout states for a given coast range. For the true missile model, all combinations of burnout states that give a coast range of 100 n mile are used to plot Fig. 2, which presents coast time vs burnout energy. Figure 2 shows that, for a given coast range, minimum coast time requires maximum burnout energy. Since the burn time (20.3 s) is small with respect to the coast time, and since the distance traveled during the thrust phase is small with respect to the target range, it is concluded that minimum total flight time to a given target implies maximum burnout energy. This result can be established for other ranges of interest.

Because of the previous discussion and the fact that the burnout mass is the same for all final conditions, the performance index is chosen to be the burnout energy per unit mass. For the true missile, the velocity part of the burnout energy is nearly two orders of magnitude higher than the altitude part (Table 1), so the performance index can be approximated by

$$J = -\tfrac{1}{2}\left(u_{bo}^2 + v_{bo}^2 + w_{bo}^2\right) \tag{20}$$

where the minus sign converts the maximization problem into a minimization problem.

Since the ballistic coast is assumed to take place in a vertical plane, the burnout velocity vector must lie in the vertical plane containing the target and the missile, that is,

$$v_{bo}(x_T - x_{bo}) - u_{bo}(y_T - y_{bo}) = 0 \tag{21}$$

The true optimal trajectories indicate that burnout occurs near launch, so that the burnout coordinates x_{bo}, y_{bo} can be replaced by the coordinates of the current sample point x_0, y_0. Hence, one final condition is given by

$$v_{bo}(x_T - x_0) - u_{bo}(y_T - y_0) = 0 \tag{22}$$

The second final condition is obtained by requiring the burnout conditions h_{bo}, V_{bo}, γ_{bo} to be such that the missile hits the target. From Table 1, it is seen that h_{bo} and V_{bo} on the average do not vary substantially with target azimuth and range, and their changes are ignored. On the other hand, γ_{bo} is a strong function of both target azimuth and range. Hence,

$$\gamma_{bo} - F(R_T, A_T) = 0 \tag{23}$$

where the function F is found by curve fitting to be

$$\begin{aligned} F = {} & 0.9542 \times 10^{-4} - 0.3684 \times 10^{-4} A_T + 0.1704 \times 10^{-4} A_T^2 \\ & - 0.1117 \times 10^{-6} A_T^3 + 0.5204 \times 10^{-2} R_T \\ & - 0.2807 \times 10^{-5} R_T A_T + 0.2787 \times 10^{-7} R_T A_T^2 \\ & - 0.2242 \times 10^{-4} R_T^2 + 0.6690 \times 10^{-8} R_T^2 A_T \\ & + 0.3825 \times 10^{-7} R_T^3 \end{aligned} \tag{24}$$

Since A_T and R_T at burnout are slightly different from the corresponding values at launch (Fig. 1), they are taken to be the values at the sample time, that is,

$$A_T = \arctan\frac{y_T - y_0}{x_T - x_0} \tag{25}$$

$$R_T = \sqrt{(x_T - x_0)^2 + (y_T - y_0)^2} \tag{26}$$

Since Eq. (22) is exact at burnout, a bounded EC parameter is introduced only into Eq. (23), and the constraint is restated in the form

$$\sin\gamma_{bo} - \sin F + e_{\gamma_{bo}} = 0 \tag{27}$$

To bound the magnitude of $e_{\gamma_{bo}}$, it is constrained indirectly by adding a weighted penalty term to the performance index (20). In terms of the state variables, Eq. (27) becomes

$$\frac{w_{bo}}{\sqrt{u_{bo}^2 + v_{bo}^2 + w_{bo}^2}} - \sin F + e_{\gamma_{bo}} = 0 \tag{28}$$

For a missile with a very high thrust-to-weight ratio, it assumed that the lift and drag terms are small with respect to the thrust and gravity terms. Hence, the aerodynamic terms are replaced by arbitrary but bounded functions of the variable of integration, which is now chosen to be the vehicle mass. In terms of these EC functions, the missile equations of motion become

$$x' = -(u/\beta) \tag{29}$$

$$y' = -(v/\beta) \tag{30}$$

$$h' = -(w/\beta) \tag{31}$$

$$u' = -(T/m\beta)\cos\theta\cos\phi + (e_u/m\beta) \tag{32}$$

$$v' = -(T/m\beta)\cos\theta\sin\phi + (e_v/m\beta) \tag{33}$$

$$w' = -(T/m\beta)\sin\theta + (g/\beta) + (e_w/m\beta) \tag{34}$$

and the controls are now the thrust azimuth and inclination. To avoid imposing bounds on the EC functions during the optimization process, they are constrained indirectly by adding a weighted penalty term to the performance index (20).

With the added EC terms, the performance index becomes

$$J = -\frac{1}{2}\left(u_{bo}^2 + v_{bo}^2 + w_{bo}^2\right) + \frac{1}{2}M_{\gamma_{bo}} e_{\gamma_{bo}}^2 + \frac{1}{2}\int_{m_0}^{m_{bo}} e^T M e \, dm \tag{35}$$

where the constant $M_{\gamma_{bo}}$ and the constant diagonal matrix M are positive EC weights.

The EC optimal control problem can now be stated as follows: Given the sample mass and states as initial conditions, find the control histories $\theta(m)$, $\phi(m)$; the EC functions $e_u(m)$, $e_v(m)$, $e_w(m)$; and the parameter $e_{\gamma_{bo}}$ that minimize the performance index (35) subject to the differential constraints (29) through (34), with the prescribed final conditions (22), (28), and m_{bo} fixed.

In general, the weights (M) can be chosen so that the values of the EC terms resulting from the optimization process roughly correspond to values expected to occur along the trajectory. However, it is better to treat the M as tuning parameters relative to the true optimization problem. Here, the EC control is used in the true optimal control problem, and the total flight time is minimized with respect to the M by nonlinear programming. What results is an improved approximate analytical optimal control.

Solution of the EC Optimal Control Problem

To solve the proposed optimal control problem,[7] the Hamiltonian and the endpoint function are written as

$$H = \frac{1}{2}M_u e_u^2 + \frac{1}{2}M_v e_v^2 + \frac{1}{2}M_w e_w^2 - \lambda_x \frac{u}{\beta} - \lambda_y \frac{v}{\beta} - \lambda_h \frac{w}{\beta} + \frac{\lambda_u}{m\beta}(-T\cos\theta\cos\phi + e_u) + \frac{\lambda_v}{m\beta}(-T\cos\theta\sin\phi + e_v) + \frac{\lambda_w}{m\beta}(-T\sin\theta + mg + e_w) \tag{36}$$

and

$$G = -\frac{1}{2}\left(u_{bo}^2 + v_{bo}^2 + w_{bo}^2\right) + \frac{1}{2}M_{\gamma_{bo}} e_{\gamma_{bo}}^2 + \nu_1\left(\frac{w_{bo}}{\sqrt{u_{bo}^2 + v_{bo}^2 + w_{bo}^2}} + e_{\gamma_{bo}} - k_1\right) + \nu_2(v_{bo} - k_2 u_{bo}) \tag{37}$$

where

$$k_1 = \sin F$$

$$k_2 = \frac{y_T - y_0}{x_T - x_0}$$

The first variation necessary conditions that must be satisfied along a minimal path are

$$X' = f(m, X, U, e) \tag{38}$$

$$\lambda' = -H_X^T(m, X, U, e, \lambda) \tag{39}$$

$$0 = H_U^T(m, X, U, \lambda) \tag{40}$$

$$0 = H_e^T(m, X, e, \lambda) \tag{41}$$

along with the boundary conditions (22), (28), and

$$\lambda_{bo} = G_{X_{bo}}^T\left(X_{bo}, \nu, e_{\gamma_{bo}}\right) \tag{42}$$

The optimal value of parameter $e_{\gamma_{bo}}$ is given by

$$G_{e_{\gamma_{bo}}} = 0 \tag{43}$$

In Eqs. (38) through (43), the state vector, the control vector, and the EC control vector are defined as

$$X = [x \quad y \quad h \quad u \quad v \quad w]^T \tag{44}$$

$$U = [\theta \quad \phi]^T \tag{45}$$

$$e = [e_u \quad e_v \quad e_w]^T \tag{46}$$

and $e_{\gamma_{bo}}$ is an unknown parameter.

The solution to the multiplier equations (39) subject to the boundary conditions (42) is that all six λ are constant; they are given by

$$\lambda_x = 0 \tag{47}$$

$$\lambda_y = 0 \tag{48}$$

$$\lambda_h = 0 \tag{49}$$

$$\lambda_u = -u_{bo} - \frac{\nu_1 w_{bo} u_{bo}}{\left(u_{bo}^2 + v_{bo}^2 + w_{bo}^2\right)^{\frac{3}{2}}} - \nu_2 k_2 \tag{50}$$

$$\lambda_v = -v_{bo} - \frac{\nu_1 w_{bo} v_{bo}}{\left(u_{bo}^2 + v_{bo}^2 + w_{bo}^2\right)^{\frac{3}{2}}} + \nu_2 \tag{51}$$

$$\lambda_w = -w_{bo} + \frac{\nu_1\left(u_{bo}^2 + v_{bo}^2\right)}{\left(u_{bo}^2 + v_{bo}^2 + w_{bo}^2\right)^{\frac{3}{2}}} \tag{52}$$

The optimality conditions (40) along with the Legendre–Clebsch condition admit the following solution:

$$\sin\theta = -\frac{\lambda_w}{\sqrt{\lambda_u^2 + \lambda_v^2 + \lambda_w^2}} \tag{53}$$

$$\cos\theta = \frac{\sqrt{\lambda_u^2 + \lambda_v^2}}{\sqrt{\lambda_u^2 + \lambda_v^2 + \lambda_w^2}} \tag{54}$$

$$\sin\phi = -\frac{\lambda_v}{\sqrt{\lambda_u^2 + \lambda_v^2}} \tag{55}$$

$$\cos\phi = -\frac{\lambda_u}{\sqrt{\lambda_u^2 + \lambda_v^2}} \tag{56}$$

The optimal values of the EC functions are determined from (41) and are given by

$$e_u = -\frac{\lambda_u}{m\beta M_u} \tag{57}$$

$$e_v = -\frac{\lambda_v}{m\beta M_v} \tag{58}$$

$$e_w = -\frac{\lambda_w}{m\beta M_w} \tag{59}$$

Finally, the optimal value of the parameter $e_{\gamma_{bo}}$ as determined from Eq. (43) is given by

$$e_{\gamma_{bo}} = -\frac{\left(u_{bo}^2 + v_{bo}^2 + w_{bo}^2\right)^{\frac{3}{2}}(\lambda_w + w_{bo})}{M_{\gamma_{bo}}\left(u_{bo}^2 + v_{bo}^2\right)} \tag{60}$$

Since $\lambda_u, \lambda_v, \lambda_w$ are constants, Eqs. (53–56) indicate that optimal controls θ and ϕ are also constant. Equations (32–34) can be integrated analytically and yield the following solutions:

$$u_{bo} = u_0 + \frac{T}{\beta}\cos\theta\cos\phi\,\ell n\frac{m_0}{m_{bo}} + \frac{\lambda_u}{\beta^2 M_u}\left(\frac{1}{m_{bo}} - \frac{1}{m_0}\right) \tag{61}$$

$$v_{bo} = v_0 + \frac{T}{\beta}\cos\theta\sin\phi\,\ell n\frac{m_0}{m_{bo}} + \frac{\lambda_v}{\beta^2 M_v}\left(\frac{1}{m_{bo}} - \frac{1}{m_0}\right) \tag{62}$$

$$w_{bo} = w_0 + \frac{T}{\beta}\sin\theta\,\ell n\frac{m_0}{m_{bo}} + \frac{g}{\beta}(m_{bo} - m_0) + \frac{\lambda_w}{\beta^2 M_w}\left(\frac{1}{m_{bo}} - \frac{1}{m_0}\right) \tag{63}$$

Combining Eqs. (50) through (52) yields

$$\lambda_u + u_{bo} + \frac{\lambda_w + w_{bo}}{A_1}A_3 + k_2\left(\lambda_v + v_{bo} - \frac{\lambda_w + w_{bo}}{A_1}\right)A_2 = 0 \tag{64}$$

Table 2 Guidance results: analytical guidance law

Test case no.	Target range, n mile	Target azimuth, deg	γ_{bo}, deg	V_{bo}, ft/s	h_{bo}, ft	Final time, s	V_f, ft/s	Miss distance, n mile
1	100	0	19.2	8097	28,354	163.7	995	−1.5
2	100	60	20.9	7865	27,864	169.0	1008	0.6
3	100	120	22.9	7511	27,826	174.5	1025	−0.2
4	100	180	16.0	7569	42,612	162.0	1003	0.2
5	200	0	25.8	8484	36,816	227.5	1507	2.0
6	200	60	28.0	8245	36,232	235.9	1610	3.5
7	200	120	31.0	7895	36,723	247.4	1755	2.9
8	200	180	27.1	7840	46,533	236.0	1659	3.7
9	300	0	33.0	8779	45,296	299.8	2665	3.2
10	300	60	36.0	8552	45,079	314.1	2840	1.9
11	300	120	41.0	8244	46,901	339.2	3131	−1.3
12	300	180	40.0	8124	52,821	335.9	3120	−1.9

where

$$A_1 = \frac{u_{bo}^2 + v_{bo}^2}{\left(u_{bo}^2 + v_{bo}^2 + w_{bo}^2\right)^{\frac{3}{2}}} \tag{65}$$

$$A_2 = -\frac{w_{bo} v_{bo}}{\left(u_{bo}^2 + v_{bo}^2 + w_{bo}^2\right)^{\frac{3}{2}}} \tag{66}$$

$$A_3 = -\frac{w_{bo} u_{bo}}{\left(u_{bo}^2 + v_{bo}^2 + w_{bo}^2\right)^{\frac{3}{2}}} \tag{67}$$

The solution of the optimal control problem reduces to the solution of three nonlinear algebraic equations in three unknowns λ_u, λ_v, and λ_w. Given values for the multipliers, Eqs. (53–56) are solved for θ and ϕ. Then, the burnout velocity components u_{bo}, v_{bo}, and w_{bo} are computed from Eqs. (61–63). Finally, values for the three equations are obtained from Eqs. (22), (28), and (64).

If the EC terms are assumed to be zero, the approximate optimal control problem can be solved analytically (see Appendix B). Since the EC solution is expected to be close to the analytical solution, λ_u, λ_v, and λ_w from the analytical solution can be used as initial guesses for the first computation of the EC solution. At all subsequent sample points, the values of λ_u, λ_v, and λ_w from the previous sample point are used as initial guesses. The values of M_u, M_v, M_w, and $M_{\gamma_{bo}}$ are chosen by using a nonlinear programming code to optimize the true performance of this guidance law. In other words, the M are computed at the launch point and held constant over the entire guided trajectory.

Midcourse Guidance Algorithm

The EC guidance law can be implemented by performing the following steps at every sample time:

1) Compute the vehicle mass and determine the vehicle state from an inertial measuring unit.

2) Solve Eqs. (22), (28), and (64) numerically for λ_u, λ_v, and λ_w.

3) Compute the values of θ and ϕ from Eqs. (53–56).

4) Compute α_0 and μ_0 using

$$\alpha = \arccos[\cos\theta\cos\gamma\cos(\phi - \psi) + \sin\theta\sin\gamma] \tag{68}$$

$$\mu = \arccos\frac{\sin\theta - \sin\gamma\cos\alpha}{\sin\alpha\cos\gamma} \tag{69}$$

and Eqs. (14) and (15). If α_0 exceeds 15 deg, use $\alpha_0 = 15$ deg and the computed value of μ_0.

5) Hold these controls constant over the sample period.

This guidance law is iterative, but only in the solution of three algebraic equations in three unknowns. Good values for the guesses are available.

Table 3 Optimal weights for EC guidance

Test case no.	Target range, n mile	Target azimuth, deg	M_u	M_v	M_w	$M_{\gamma_{bo}}$
1	100	0	0.114	N/A	0.035	1.61×10^{12}
2	100	60	0.504	0.506	0.037	3.95×10^{10}
3	100	120	0.056	0.049	0.030	2.93×10^{13}
4	100	180	0.278	0.245	0.104	1.10×10^{14}
5	200	0	0.090	N/A	0.040	4.02×10^{11}
6	200	60	0.035	0.265	0.014	1.36×10^{10}
7	200	120	0.030	0.030	0.024	2.77×10^{11}
8	200	180	0.205	0.219	0.067	2.01×10^{10}
9	300	0	0.091	N/A	0.040	4.75×10^{11}
10	300	60	0.075	0.075	0.034	1.54×10^{11}
11	300	120	0.046	0.046	0.024	1.43×10^{11}
12	300	180	0.310	0.246	0.100	2.60×10^{11}

Numerical Results

To evaluate the performances of the analytical guidance law and the EC guidance law, twelve test cases have been selected. The missile is targeted to a point 1000 ft above the target with the anticipation that midcourse guidance will switch to homing guidance before the missile actually reaches that point. The guidance interval is assumed to be 0.5 s.

Table 2 is a summary of the results obtained by the analytical guidance law (no EC terms). Note that the flight times and the final velocities are close to the true optimal values, but that miss distances on the order of 4 n mile occur. However, if the missile is handed over to terminal guidance (PRONAV) prior to impact, the miss distances can be reduced to feet.[8]

For the EC guidance law, the tuning parameters M_u, M_v, M_w, and $M_{\gamma_{bo}}$ have been obtained for an array of target locations and are presented in Table 3. Once a sufficient number of solutions are obtained, optimal weights can be tabulated as functions of the target range and target azimuth for real-time onboard implementation.

Results obtained with the EC guidance law are summarized in Table 4. While the flight times and the final velocities do not differ much from those of the analytical guidance law, the missile gets within 10^{-2} n mile of the target without terminal guidance.

Results of test case 6 are presented graphically. Ground tracks of the guided trajectories and the true optimal trajectories are compared in Fig. 3. Note that the guided trajectories make a sharper turn than the true optimal trajectories. This is because the guidance laws command a higher bank angle. Figure 4 shows the altitude profiles of the guided trajectories and the true optimal trajectories.

Note that the EC guidance law does not change the missile trajectory very much from the analytical guidance law, but it does make the missile hit the target. Without the EC parameter on γ_{bo}, the EC guidance law does not hit the target. However, because the EC functions improve the trajectory somewhat and because the EC parameter alone makes the guidance law iterative, all four EC terms are included in the guidance law.

Table 4 Guidance results: EC guidance law

Test case no.	Target range, n mile	Target azimuth, deg	γ_{bo}, deg	V_{bo}, ft/s	h_{bo}, ft	Final time, s	V_f, ft/s	Miss distance, n mile
1	100	0	19.2	8091	30,023	165.8	970	2.0×10^{-3}
2	100	60	21.0	7823	28,930	170.4	979	1.2×10^{-3}
3	100	120	23.0	7496	29,025	175.9	992	8.0×10^{-3}
4	100	180	16.0	7571	42,201	162.1	1003	1.3×10^{-3}
5	200	0	25.8	8442	37,995	227.9	1371	1.6×10^{-3}
6	200	60	28.3	8183	36,840	236.3	1465	1.3×10^{-3}
7	200	120	31.0	7862	37,575	246.9	1593	1.9×10^{-3}
8	200	180	27.2	7846	45,838	235.9	1495	2.7×10^{-3}
9	300	0	33.0	8731	46,410	299.3	2447	4.0×10^{-3}
10	300	60	36.0	8520	46,330	314.0	2637	2.3×10^{-4}
11	300	120	41.0	8244	48,471	340.7	2967	1.1×10^{-3}
12	300	180	40.0	8140	53,092	337.0	3145	1.2×10^{-3}

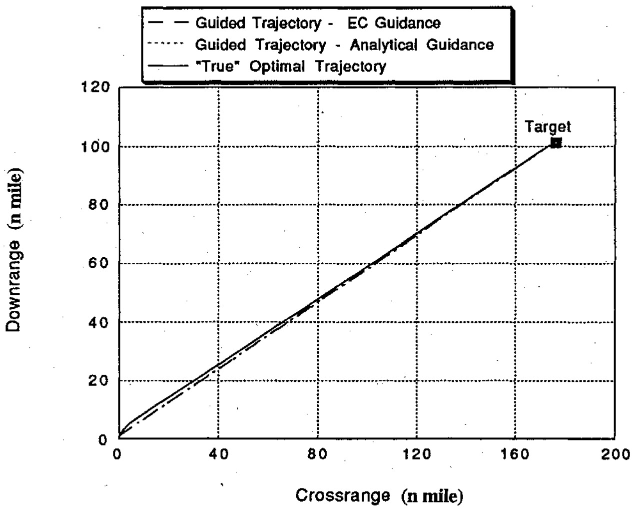

Fig. 3 Ground track comparison: test case 6.

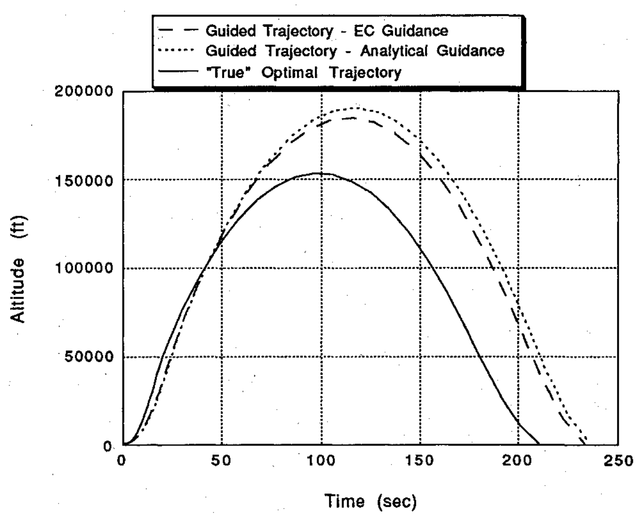

Fig. 4 Altitude profile comparison: test case 6.

Discussion and Conclusions

An optimal guidance law is developed for a high-thrust, bank-to-turn SRAM. Its mission is assumed to be the following: The missile is launched subsonically at low altitude from an aircraft; it performs a thrusting, climbing turn to point toward the target at burnout; and it coasts in a vertical plane at zero angle of attack to a fixed target point. The target can be located between 100 and 300 n mile from the launch aircraft and at any point on the compass.

First, true optimal trajectories are computed assuming piecewise linear controls and applying an available nonlinear programming code. Second, an approximate analytical guidance law is derived by formulating the optimal control problem over just the thrust phase. Aerodynamic terms are neglected relative to the thrust terms, and approximate final conditions are created so that the missile hits the fixed target point at the end of the ballistic coast. In simulation, this guidance law works well for midcourse guidance, but the missile has a maximum miss distance of 4 n mile, which can be eliminated by terminal guidance. Third, the analytical guidance law is improved by introducing error compensation terms to replace the neglected aerodynamic and final condition terms. The EC guidance law produces flight times within 15% of the true optimal values and miss distances within 10^{-2} n mile. The weights associated with the EC guidance law are computed numerically by using the EC control in the true optimal control problem. Hence, what results is an improved approximate analytical guidance law.

Table A1 Aerodynamic data

M	$C_{L\alpha}$	C_{D0}	K
0.0	8.18	0.275	0.145
0.8	8.18	0.251	0.130
1.2	8.88	0.506	0.126
1.5	8.59	0.460	0.127
2.0	8.30	0.405	0.141
3.0	7.36	0.309	0.167
4.0	6.53	0.244	0.197
6.0	5.72	0.190	0.229
8.0	5.72	0.154	0.241
20	5.72	0.154	0.241

Appendix A: Missile Characteristics

The thrust, mass, and aerodynamic characteristics of a SRAM are

$$T = 20{,}860 \text{ lb}, \qquad \beta = 2.38 \text{ slugs/s}, \qquad m_{\text{launch}} = 68.3 \text{ slugs}$$

$$m_{bo} = 19.9 \text{ slugs}, \qquad S = 2.18 \text{ ft}^2$$

and $C_{L\alpha}(M)$, $C_{D0}(M)$, and $K(M)$ are shown in Table A1.

Appendix B: Analytical Guidance Law

If the EC terms are assumed to be zero, the approximate optimal control problem can be solved analytically.[1] The optimal controls are given by

$$\theta = \arcsin \frac{-b + \sqrt{b^2 - 4ad}}{2a}$$

$$\phi = \arccos \frac{c_1 + c_2 \sin\theta}{k_3 c_2 \cos\theta}$$

where

$$k_1 = \sin F$$

$$k_2 = \frac{y_T - y_0}{x_T - x_0}$$

$$k_3 = \sqrt{k_1^2 \frac{1 + k_2^2}{1 - k_1^2}}$$

$$k_4 = (k_2/k_3)$$

$$c_1 = w_0 - k_3 u_0 - (g/\beta)(m_0 - m_{bo})$$

$$c_2 = \frac{T}{\beta} \ell n \frac{m_0}{m_{bo}}$$

$$c_3 = k_4[w_0 - (g/\beta)(m_0 - m_{bo})] - v_0$$

$$a = c_2^2 + k_4^2 k_3^2 c_2^2 + c_2^2 k_3^2$$

$$b = 2c_1 c_2 + 2k_3^2 c_3 k_4 c_2$$

$$d = c_1^2 + c_3^2 k_3^2 - k_3^2 c_2^2$$

The burnout velocity components are given by

$$u_{bo} = u_0 + c_2 \cos\theta \cos\phi$$

$$v_{bo} = v_0 + c_2 \cos\theta \sin\phi$$

$$w_{bo} = w_0 + c_2 \sin\theta + (g/\beta)(m_{bo} - m_0)$$

The Lagrange multipliers are

$$\lambda_x = 0$$

$$\lambda_y = 0$$

$$\lambda_h = 0$$

$$\lambda_u = \frac{-u_{bo}A_1 + w_{bo}A_3 - k_2 v_{bo} A_1 + k_2 A_2 w_{bo}}{A_1 - c_4 A_3 + A_1 k_2 \tan\phi + c_4 k_2 A_2}$$

$$\lambda_v = \lambda_u \tan\phi$$

$$\lambda_w = \lambda_u c_4$$

where

$$c_4 = \tan\theta\cos\phi + \frac{\tan\theta\sin^2\phi}{\cos\phi}$$

and A_1, A_2, and A_3 are given by Eqs. (65–67). The constant Lagrange multipliers are given by

$$\nu_1 = \frac{\lambda_w + w_{bo}}{A_1}$$

$$\nu_2 = \lambda_v + v_{bo} - \nu_1 A_2$$

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F. H. Lutze
Associate Editor